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Lifting some approximation properties from a dual space X' to the Banach space X

by

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Dedicated to the memory of Eve Oja

Abstract. For a fixed Banach operator ideal $\mathcal{A}$, we characterize $\mathcal{A}$-compact sets (in the sense of Carl and Stephani) that are determined by c_0 via the Banach composition ideal $\mathcal{A}\circ\mathfrak{K}_\infty$, with $\mathfrak{K}_\infty$ the Banach ideal of Fourie and Swart. This characterization allows us to relate $\mathcal{K}_\mathcal{A}$-approximation properties on a Banach space and $\mathcal{K}_\mathcal{B}$-approximation properties on its dual space, where $\mathcal{A}$ and $\mathcal{B}$ are ideals linked by some classical procedures. These approximation properties have been widely studied in several papers in the last years.

Introduction. A Banach space X has the approximation property (AP) if the identity map of X can be approximated by finite rank operators over compact sets. Since Enflo constructed the first example of a Banach space failing to have the AP [4], many different notions of AP have emerged. One of those variants, which has received the attention of several authors in the last years, consists in studying when the identity map can be approximated (in some sense) by finite rank operators over *some* compact sets. A very useful tool to describe *some* compact sets is the theory of $\mathcal{A}$-compact sets of Carl and Stephani [1], where $\mathcal{A}$ denotes a Banach operator ideal.

It is known from Grothendieck's memoirs that the AP passes from a dual space to its underlying space [6, Proposition 36]. This result motivated the study of different situations for which an AP in a dual space determines another kind of AP in the space. In this short note we are focused on approximation properties determined by certain $\mathcal{A}$-compact sets. The question we deal with can be framed as follows.

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Question. Let X be a Banach space with dual space X'. Suppose that the identity map of X' is approximated by finite rank operators over a class of compact sets. Over which class of compact sets, can the identity map of X be approximated by finite rank operators?

This question was already answered for particular $\mathcal{A}$-compact sets in several papers, see [2, 7, 8, 9, 11] among others. Our objective is to study the above problem for a wide class of $\mathcal{A}$-compact sets. Our main results are Theorems 2.4 and 2.5.

In [11, Proposition 1.8], it was shown that $\mathcal{A}$-compact sets are determined by operators in $\mathcal{A}$ with domain in ℓ_1. Since Theorems 2.4 and 2.5 involve $\mathcal{A}$-compact sets determined by operators with domain c_0, we make a brief discussion of this type of compact sets in Section 1. In Section 2 we discuss results on lifting approximation properties.

We will need some basics of the theory of operator ideals and tensor norms. All the details we use can be found in the books of Pietsch [14] (for operator ideals) and of Defant and Floret [3] (for tensor norms). For the basics of the theory of approximation properties, we refer the reader to the book of Lindenstrauss and Tzafriri [12].

1. Preliminaries. As usual, X, Y denote Banach spaces over the same field $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. The closed unit ball of X and its dual space are denoted by B_X and X' respectively. Let $\mathcal{A}$ be a Banach operator ideal. Following Carl and Stephani [1], we call a subset K of X relatively $\mathcal{A}$-*compact* if there exist a Banach space Z, an operator $T \in \mathcal{A}(Z, X)$ and a compact subset L of Z such that $K \subset T(L)$. The measure of relatively $\mathcal{A}$-compact sets was introduced in [11] as follows. Given a relatively $\mathcal{A}$-compact subset K of X, the size of K is given by

$$m_{\mathcal{A}}(K; X) := \inf\{\|T\|_{\mathcal{A}} \colon T \in \mathcal{A}(Z, X),\ L \subset B_Z \text{ compact},\ K \subset T(L)\}.$$

From [11, Proposition 1.8], we may replace any Banach space Z by ℓ_1. A linear map $R\colon Y \to X$ is said to be $\mathcal{A}$-*compact* if $R(B_Y)$ is a relatively $\mathcal{A}$-compact subset of X. Let $\mathcal{K}_{\mathcal{A}}(Y, X)$ denote the space of all $\mathcal{A}$-compact operators from Y to X. It becomes a Banach operator ideal when endowed with the norm $\|R\|_{\mathcal{K}_{\mathcal{A}}} := m_{\mathcal{A}}(R(B_Y); X)$ for $R \in \mathcal{K}_{\mathcal{A}}(Y, X)$.

For a Banach operator ideal $\mathcal{A}$, we will use $\mathcal{A}^{\max}$, $\mathcal{A}^{\min}$, $\mathcal{A}^{\mathrm{sur}}$, $\mathcal{A}^{\mathrm{inj}}$, $\mathcal{A}^{\mathrm{reg}}$, $\mathcal{A}^{\mathrm{dual}}$ and $\mathcal{A}^{\mathrm{adj}}$ to denote the maximal, minimal, surjective, injective, regular, dual and adjoint Banach operator ideals, respectively. The definitions can be found in [14, Section 8], except for $\mathcal{A}^{\mathrm{adj}}$ which is defined in [14, 9.1.2]. We consider only finitely generated tensor norms [3, 12.4]. If α is a tensor norm, α^t, α', $\backslash\alpha$ and $/\alpha$ denote, respectively, the transpose, dual, left-projective and left-injective tensor norms associated with α. See [3, 12.3], [3, 15.2], [3, 20.6] and [3, 20.7] for their definitions.

A Banach operator ideal $\mathcal{A}$ and a tensor norm α are *associated* if for all finite-dimensional spaces M, N the equality $\mathcal{A}(M,N) = M' \otimes_\alpha N$ holds isometrically. From [3, 22.1] we know that $\mathcal{A}$, $\mathcal{A}^{\min}$ and $\mathcal{A}^{\max}$ are associated with the same tensor norm. In light of [11], the theory of $\mathcal{A}$-compactness is simpler whenever $\mathcal{A}$ is a right-accessible Banach operator ideal. This happens when the identity $\mathcal{A}^{\min} = \mathcal{A} \circ \overline{\mathcal{F}}$ holds isometrically. It is worth mentioning that most of the known ideals are right-accessible. As usual, $\overline{\mathcal{F}}$ and $\mathcal{L}$ are the ideals of approximable and continuous operators, respectively; considered with the supremum norm.

As commented before, we will appeal to $\mathcal{A}$-compact sets determined by operators with domain in c_0. To characterize and place them in the framework of the Carl and Stephani theory, we will use the Banach ideal $\mathfrak{K}_\infty$ of Fourie and Swart [5]. An operator $T \in \mathcal{L}(X,Y)$ belongs to $\mathfrak{K}_\infty$ if there exist approximable operators $S \in \overline{\mathcal{F}}(X, c_0)$ and $R \in \overline{\mathcal{F}}(c_0, Y)$ such that $T = RS$; the norm of T is given by $\|T\|_{\mathfrak{K}_\infty} := \inf\{\|R\|\,\|S\| \colon T = RS\}$, where the infimum is taken over all factorizations of T through c_0 by approximable operators. Recall that for a Banach operator ideal $\mathcal{A}$, the composition ideal $\mathcal{A} \circ \mathfrak{K}_\infty$ is a Banach operator ideal endowed with the composition norm. Now, we have the following characterization.

PROPOSITION 1.1. *Let $\mathcal{A}$ be a Banach operator ideal, X a Banach space and $K \subset X$ a subset. Then K is relatively $\mathcal{A} \circ \mathfrak{K}_\infty$-compact if and only if there exist $T \in \mathcal{A}(c_0, X)$ and a compact set $L \subset B_{c_0}$ such that $K \subset T(L)$. Moreover,*

$$m_{\mathcal{A}\circ\mathfrak{K}_\infty}(K;X) = \inf\{\|T\|_{\mathcal{A}} \colon T \in \mathcal{A}(c_0, X),\ L \subset B_{c_0} \text{ compact},\ K \subset T(L)\}.$$

Proof. Suppose that K is a $\mathcal{A} \circ \mathfrak{K}_\infty$-compact and take $\varepsilon > 0$. There exist an operator $R \in \mathcal{A} \circ \mathfrak{K}_\infty(\ell_1, X)$ and a relatively compact subset $\widetilde{L}$ of B_{ℓ_1} such that $K \subset R(\widetilde{L})$ and

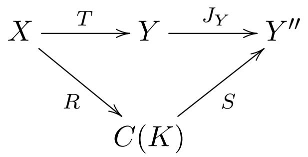
$$\|R\|_{\mathcal{A}\circ\mathfrak{K}_\infty} \le (1+\varepsilon) m_{\mathcal{A}\circ\mathfrak{K}_\infty}(K;X).$$

Since $R \in \mathcal{A} \circ \mathfrak{K}_\infty(\ell_1, X)$, there exist a Banach space Z and operators $R_1 \in \overline{\mathcal{F}}(\ell_1, c_0)$, $R_2 \in \overline{\mathcal{F}}(c_0, Z)$ and $R_3 \in \mathcal{A}(Z, X)$ such that $R = R_3 R_2 R_1$ with $\|R_2 R_1\| = 1$ and $\|R_3\|_{\mathcal{A}} \le (1+\varepsilon)\|R\|_{\mathcal{A}\circ\mathfrak{K}_\infty}$. We may assume that $\|R_2\| = 1$ and $\|R_1\| \le 1+\varepsilon$. Furthermore, rescaling R_3 and R_1, we may assume that $\|R_1\| \le 1$ and $\|R_3\|_{\mathcal{A}} \le (1+\varepsilon)^2 \|R\|_{\mathcal{A}\circ\mathfrak{K}_\infty}$. Denoting $L = R_1(\widetilde{L})$ and $T = R_3 R_2$ we see that $L \subset B_{c_0}$ is a compact set, $T \in \mathcal{A}(c_0, X)$ and $K \subset T(L)$ with

$$\|T\|_{\mathcal{A}} \le (1+\varepsilon)\|R\|_{\mathcal{A}\circ\mathfrak{K}_\infty} \le (1+\varepsilon)^3 m_{\mathcal{A}\circ\mathfrak{K}_\infty}(K;X).$$

For the converse, suppose that $K \subset T(L)$ with $T \in \mathcal{A}(c_0, X)$ and $L \subset B_{c_0}$ compact. In particular, L is $\overline{\mathcal{F}}$-compact [10, Remark 1.3], which implies that there exist an approximable operator $R \in \overline{\mathcal{F}}(\ell_1, c_0)$ with $\|R\| \le 1$ and a compact set $\widetilde{L} \subset B_{\ell_1}$ such that $L \subset R(\widetilde{L})$. By [5, Corollary 2.4], $R \in \mathfrak{K}_\infty(\ell_1, c_0)$ with $\|R\|_{\mathfrak{K}_\infty} \le 1$, implying that $K \subset TR(\widetilde{L})$. Since $TR \in \mathcal{A} \circ \mathfrak{K}_\infty(\ell_1, X)$, the conclusion follows. ∎

To describe the tensor norm associated with $\mathcal{A} \circ \mathfrak{K}_\infty$ we will make use of the ideal of ∞-factorable operators, denoted by $\mathcal{L}_\infty$. Recall that $\mathcal{L}_\infty$ is maximal and $\mathcal{L}_\infty^{\min} = \mathfrak{K}_\infty$ [14, Theorems 19.1.1 and 19.1.2]. By [14, 19.3.9], $T \in \mathcal{L}_\infty(X, Y)$ if and only if there exist a compact Hausdorff space K and operators $R \in \mathcal{L}(X, C(K))$ and $S \in \mathcal{L}(C(K), Y'')$ such that the following diagram is commutative:



where $J_Y \colon Y \to Y''$ is the natural inclusion of Y into its bidual and $C(K)$ is the Banach space of all continuous functions from K to $\mathbb{K}$. In this case, $\|T\|_{\mathcal{L}_\infty} := \inf\{\|R\| \, \|S\|\}$, where the infimum is taken over all such factorizations. It is straightforward from its definition that $\mathcal{L}_\infty$ is regular, that is, $\mathcal{L}_\infty = (\mathcal{L}_\infty)^{\mathrm{reg}}$. Also, we will need the following.

Lemma 1.2 (Oertel). *Let $\mathcal{A}$ be a maximal Banach operator ideal associated with the tensor norm α. Then, the maximal Banach operator ideal associated with $\backslash\alpha$ is $(\mathcal{A} \circ \mathcal{L}_\infty)^{\mathrm{reg}}$ and $(\mathcal{A} \circ \mathcal{L}_\infty)^{\mathrm{reg}} = ((\mathcal{A}^{\mathrm{adj}})^{\mathrm{inj}})^{\mathrm{adj}}$.*

Proof. Since $\mathcal{A}$ is maximal, $\mathcal{A} = (\mathcal{A}^{\mathrm{adj}})^{\mathrm{adj}}$. Now, the proof corresponds to the paragraph that precedes [13, Lemma 4.1]. ∎

Proposition 1.3. *Let $\mathcal{A}$ be a Banach operator ideal associated with a tensor norm α. Then the Banach operator ideal $\mathcal{A} \circ \mathfrak{K}_\infty$ is associated with $\backslash\alpha$.*

Proof. As $\mathcal{L}_\infty$ is right-accessible, using [3, p. 334] we get

$$(\mathcal{A}^{\max} \circ \mathcal{L}_\infty)^{\min} = \mathcal{A}^{\min} \circ \mathfrak{K}_\infty \subset \mathcal{A} \circ \mathfrak{K}_\infty \subset \mathcal{A}^{\max} \circ \mathcal{L}_\infty \subset (\mathcal{A}^{\max} \circ \mathcal{L}_\infty)^{\mathrm{reg}}.$$

Since on finite-dimensional spaces the minimal kernel of a Banach operator ideal and its regular hull coincide, we see that $(\mathcal{A}^{\max} \circ \mathcal{L}_\infty)^{\min}$ and $(\mathcal{A}^{\max} \circ \mathcal{L}_\infty)^{\mathrm{reg}}$ (and therefore $\mathcal{A} \circ \mathfrak{K}_\infty$) are associated with the same tensor norm. The conclusion follows by Oertel's result, Lemma 1.2. ∎

Notice that from [3, Theorem 20.11(2)], a maximal Banach operator ideal is surjective if and only if its associated tensor norm is left-injective. That is, for a maximal Banach operator ideal $\mathcal{A}$ associated with the tensor norm α, $\mathcal{A} = \mathcal{A}^{\mathrm{sur}}$ isometrically if and only if $\alpha = /\alpha$. We finish this section with the following proposition that will be used later.

Proposition 1.4. *Let $\mathcal{A}$ be a Banach operator ideal associated with a norm α. The following statements are equivalent:*

(a) $(\mathcal{A}^{\mathrm{adj}})^{\mathrm{dual}}$ *is surjective.*
(b) *The tensor norm α is left-projective.*
(c) $\mathcal{A}^{\min} = \mathcal{A}^{\min} \circ \mathfrak{K}_\infty$.

In particular, if $\mathcal{A}$ is right-accessible, then (c) *reads* $\mathcal{A}^{\min} = \mathcal{A} \circ \mathfrak{K}_\infty$.

Proof. To see that (a) and (b) are equivalent, we first note that if $\mathcal{A}$ is associated with α, a combination of [3, 17.8] and [3, 17.9] shows that α' is associated with $(\mathcal{A}^{\mathrm{adj}})^{\mathrm{dual}}$. Thus, since $(\mathcal{A}^{\mathrm{adj}})^{\mathrm{dual}}$ is maximal, by [3, Theorem 20.11(2)], $(\mathcal{A}^{\mathrm{adj}})^{\mathrm{dual}}$ is surjective if and only if $/(\alpha') = \alpha'$. As $\alpha = \alpha''$ [3, 15.3], using a left version of [3, Proposition 20.10] we find that $/(\alpha') = \alpha'$ if and only if $\alpha = (/(\alpha'))' = \backslash(\alpha'') = \backslash\alpha$, that is, if and only if α is left-projective.

Let us see that (b) implies (c). If $\alpha = \backslash\alpha$, Proposition 1.3 implies that $\mathcal{A}^{\max} = (\mathcal{A} \circ \mathfrak{K}_\infty)^{\max}$. Thus, $\mathcal{A}^{\min} = (\mathcal{A} \circ \mathfrak{K}_\infty)^{\min} = \mathcal{A}^{\min} \circ \mathfrak{K}_\infty$.

Finally, suppose that (c) holds. As $\mathcal{A}^{\min} = \mathcal{A}^{\min} \circ \mathfrak{K}_\infty$, proceeding as in the proof of Proposition 1.3 we obtain $\mathcal{A}^{\min} = (\mathcal{A}^{\max} \circ \mathcal{L}_\infty)^{\min}$. Thus, $\mathcal{A}^{\max} = (\mathcal{A}^{\max} \circ \mathcal{L}_\infty)^{\max}$, which implies that the associated tensor norms of $\mathcal{A}$ and $\mathcal{A}^{\max} \circ \mathcal{L}_\infty$ (and therefore those associated to $\mathcal{A}^{\max}$ and $(\mathcal{A}^{\max} \circ \mathcal{L}_\infty)^{\mathrm{reg}}$) coincide. Then, an application of Lemma 1.2 shows that $\alpha = \backslash\alpha$ and (b) is satisfied. ∎

2. Duality. We now focus on a class of approximation properties determined by $\mathcal{A}$-compact sets. Following [11, Proposition 3.1], a Banach space X has the $\mathcal{K}_\mathcal{A}$-*approximation property* ($\mathcal{K}_\mathcal{A}$-AP for short) if, given $\varepsilon > 0$ and an $\mathcal{A}$-compact set $K \subset X$, there exists a finite rank operator $S\colon X \to X$ such that

$$\mathfrak{m}_\mathcal{A}((S - \mathrm{Id}_X)(K); X) \le \varepsilon.$$

We will use and also extend the following result [7, Corollary 3.4].

Proposition 2.1 (Kim). *Let $\mathcal{A}$ be a maximal right-accessible Banach operator ideal such that $(\mathcal{A}^{\mathrm{adj}})^{\mathrm{dual}}$ is surjective and let X be a Banach space. If X' has the $\mathcal{K}_{(\mathcal{A}^{\mathrm{adj}})^{\mathrm{dual}}}$-AP, then X has the $\mathcal{K}_\mathcal{A}$-AP.*

Before presenting our main result, we give the following lemma.

Lemma 2.2. *Let $\mathcal{A}$ be a Banach operator ideal. Then*

$$\mathcal{K}_{(\mathcal{A}^{\mathrm{adj}})^{\mathrm{dual}}} = \mathcal{K}_{((\mathcal{A} \circ \mathfrak{K}_\infty)^{\mathrm{adj}})^{\mathrm{dual}}} \quad \textit{isometrically.}$$

Proof. Let α be the tensor norm associated with $\mathcal{A}$. Applying Proposition 1.3, we know that $((\mathcal{A} \circ \mathfrak{K}_\infty)^{\mathrm{adj}})^{\mathrm{dual}}$ is associated with

$$(\backslash\alpha)' = /\alpha',$$

and, in turn, $/\alpha'$ is associated with $((\mathcal{A}^{\mathrm{adj}})^{\mathrm{dual}})^{\mathrm{sur}}$. Consequently, as both are maximal ideals, we have $((\mathcal{A}\circ\mathfrak{K}_\infty)^{\mathrm{adj}})^{\mathrm{dual}} = (\mathcal{A}^{\mathrm{adj\,dual}})^{\mathrm{sur}}$. Then, in view of [1, Theorem 2.1],

$$\mathcal{K}_{((\mathcal{A}\circ\mathfrak{K}_\infty)^{\mathrm{adj}})^{\mathrm{dual}}} = \mathcal{K}_{((\mathcal{A}^{\mathrm{adj}})^{\mathrm{dual}})^{\mathrm{sur}}} = \mathcal{K}_{(\mathcal{A}^{\mathrm{adj}})^{\mathrm{dual}}},$$

and the result is proved. ∎

LEMMA 2.3. *Let $\mathcal{A}$ be a maximal right-accessible Banach operator ideal. Then $(\mathcal{A}\circ\mathfrak{K}_\infty)^{\max}$ is also right-accessible.*

Proof. Notice that by Proposition 1.3 and Lemma 1.2, we have $(\mathcal{A}\circ\mathfrak{K}_\infty)^{\max} = ((\mathcal{A}^{\mathrm{adj}})^{\mathrm{inj}})^{\mathrm{adj}}$, both sides being maximal ideals. Now, if $\mathcal{A}$ is maximal and right-accessible, combining [3, Corollary 21.3] with [3, Ex. 21.1(d)] we infer in particular that $(\mathcal{A}^{\mathrm{adj}})^{\mathrm{inj}}$ is left-accessible. Applying again [3, Corollary 21.3] we find that $((\mathcal{A}^{\mathrm{adj}})^{\mathrm{inj}})^{\mathrm{adj}}$ is right-accessible, and the result is proved. ∎

THEOREM 2.4. *Let $\mathcal{A}$ be a maximal right-accessible Banach operator ideal and X be a Banach space. If X' has the $\mathcal{K}_{(\mathcal{A}^{\mathrm{adj}})^{\mathrm{dual}}}$-AP, then X has the $\mathcal{K}_{\mathcal{A}\circ\mathfrak{K}_\infty}$-AP.*

Proof. Set $\mathcal{B} = (\mathcal{A}\circ\mathfrak{K}_\infty)^{\max}$, which by Lemma 2.3 is right-accessible. Since $\mathfrak{K}_\infty = \mathfrak{K}_\infty\circ\mathfrak{K}_\infty$ isometrically, it can be seen that $\mathcal{B}^{\min} = \mathcal{B}^{\min}\circ\mathfrak{K}_\infty$. Thus, by Proposition 1.4, $(\mathcal{B}^{\mathrm{adj}})^{\mathrm{dual}}$ is surjective. An application of Proposition 2.1 to $\mathcal{B}$ gives the $\mathcal{K}_{(\mathcal{A}\circ\mathfrak{K}_\infty)^{\max}}$-AP for X whenever X' has the $\mathcal{K}_{((\mathcal{A}\circ\mathfrak{K}_\infty)^{\mathrm{adj}})^{\mathrm{dual}}}$-AP.

Now, on the one hand, Lemma 2.2 states that $\mathcal{K}_{((\mathcal{A}\circ\mathfrak{K}_\infty)^{\mathrm{adj}})^{\mathrm{dual}}} = \mathcal{K}_{(\mathcal{A}^{\mathrm{adj}})^{\mathrm{dual}}}$. On the other hand, as $(\mathcal{A}\circ\mathfrak{K}_\infty)^{\max}$ is right-accessible, by [11, Theorem 2.11] we have the identity $\mathcal{K}_{(\mathcal{A}\circ\mathfrak{K}_\infty)^{\max}} = \mathcal{K}_{(\mathcal{A}\circ\mathfrak{K}_\infty)^{\min}} = \mathcal{K}_{\mathcal{A}\circ\mathfrak{K}_\infty}$. Thus, the proof is complete. ∎

Notice that the above theorem extends [7, Corollary 3.4]. Indeed, suppose that in addition to the hypothesis on $\mathcal{A}$ in Theorem 2.4, $(\mathcal{A}^{\mathrm{adj}})^{\mathrm{dual}}$ is surjective. Then, by Proposition 1.4, $\mathcal{A}^{\min} = \mathcal{A}\circ\mathfrak{K}_\infty$ and by [10, Proposition 1.2], $\mathcal{K}_{\mathcal{A}^{\min}} = \mathcal{K}_{\mathcal{A}}$.

Finally, we have the following.

THEOREM 2.5. *Let $\mathcal{A}$ be a maximal right-accessible Banach operator ideal and let X be a Banach space. If X' has the $\mathcal{K}_{\mathcal{A}}$-AP, then X has the $\mathcal{K}_{(\mathcal{A}^{\mathrm{adj}})^{\mathrm{dual}}\circ\mathfrak{K}_\infty}$-AP.*

Proof. Since $\mathcal{A}$ is maximal, by [3, Corollary 21.3] we know that $\mathcal{A}$ is right-accessible if and only if $(\mathcal{A}^{\mathrm{adj}})^{\mathrm{dual}}$ is right-accessible. On the other hand, the maximality of $\mathcal{A}$ also gives $(((\mathcal{A}^{\mathrm{adj}})^{\mathrm{dual}})^{\mathrm{adj}})^{\mathrm{dual}} = \mathcal{A}$. Now, the result follows by applying Theorem 2.4 to the ideal $(\mathcal{A}^{\mathrm{adj}})^{\mathrm{dual}}$. ∎

To conclude this note, we will show that the above result, applied to appropriate ideals and combined with the classical procedures, allows us to recover [11, Theorem 4.7] and [8, Theorem 1.1]. In order to see this we will use the ideals of nuclear and integral operators, denoted as usual by $\mathcal{N}$ and $\mathcal{I}$, respectively. Recall that $\mathcal{N}^{\max} = \mathcal{I}$ and $\mathcal{I}^{\min} = \mathcal{N}$. Also, we will use the fact that when $\mathcal{A}$ is a right-accessible Banach operator ideal $\mathcal{K}_{\mathcal{A}} = (\mathcal{A} \circ \overline{\mathcal{F}})^{\mathrm{sur}} = \mathcal{K}_{\mathcal{A}^{\min}}$ (see [10, Proposition 2.1]).

Corollary 2.6 ([11, Theorem 4.7]). *Let X be a Banach space. If X' has the $\mathcal{K}_{\mathcal{N}}$-AP, then X has the $\mathcal{K}_{\mathfrak{K}_\infty}$-AP.*

Proof. Taking $\mathcal{A} = \mathcal{I}$ in Theorem 2.5, we find that if X' has the $\mathcal{K}_{\mathcal{I}}$-AP, then X has the $\mathcal{K}_{(\mathcal{I}^{\mathrm{adj}})^{\mathrm{dual}} \circ \mathfrak{K}_\infty}$-AP. Let us show that this is all we need. First, since $\mathcal{I}$ is right-accessible, we have $\mathcal{K}_{\mathcal{I}} = \mathcal{K}_{\mathcal{N}}$. Now, as $\mathcal{I}^{\mathrm{adj}} = \mathcal{L}$, we have $(\mathcal{I}^{\mathrm{adj}})^{\mathrm{dual}} \circ \mathfrak{K}_\infty = \mathfrak{K}_\infty$, and the conclusion follows. ∎

Corollary 2.7 ([8, Theorem 1.1]). *Let X be a Banach space. If X' has the $\mathcal{K}_{\mathfrak{K}_\infty}$-AP, then X has the $\mathcal{K}_{\mathcal{N}}$-AP.*

Proof. Applying Theorem 2.5 to the ideal $\mathcal{L}_\infty$ we find that if X' has the $\mathcal{K}_{\mathcal{L}_\infty}$-AP, then X has the $\mathcal{K}_{(\mathcal{L}_\infty^{\mathrm{adj}})^{\mathrm{dual}} \circ \mathfrak{K}_\infty}$-AP. Again, let us show that this yields the claim. First, since $\mathcal{L}_\infty$ is right-accessible and $\mathcal{L}_\infty^{\min} = \mathfrak{K}_\infty$, we have $\mathcal{K}_{\mathcal{L}_\infty} = \mathcal{K}_{\mathfrak{K}_\infty}$. It remains to see that $\mathcal{K}_{(\mathcal{L}_\infty^{\mathrm{adj}})^{\mathrm{dual}} \circ \mathfrak{K}_\infty} = \mathcal{K}_{\mathcal{N}}$. First, since $\mathcal{N} \subset \mathcal{A}$ for any Banach operator ideal $\mathcal{A}$, we have $\mathcal{K}_{\mathcal{N}} \subset \mathcal{K}_{(\mathcal{L}_\infty^{\mathrm{adj}})^{\mathrm{dual}} \circ \mathfrak{K}_\infty}$. For the other inclusion, we apply Lemma 1.2 to $\mathcal{L}$ and use $\mathcal{L}^{\mathrm{adj}} = \mathcal{I}$, thus $\mathcal{L}_\infty = \mathcal{L}_\infty^{\mathrm{reg}} = (\mathcal{I}^{\mathrm{inj}})^{\mathrm{adj}}$. Then, $(\mathcal{L}_\infty^{\mathrm{adj}})^{\mathrm{dual}} = (((\mathcal{I}^{\mathrm{inj}})^{\mathrm{adj}})^{\mathrm{adj}})^{\mathrm{dual}} = (\mathcal{I}^{\mathrm{inj}})^{\mathrm{dual}}$. Now, appealing to [14, 8.5.9], we see that $(\mathcal{I}^{\mathrm{inj}})^{\mathrm{dual}} = (\mathcal{I}^{\mathrm{dual}})^{\mathrm{sur}} = \mathcal{I}^{\mathrm{sur}}$. Since $\mathcal{I}^{\mathrm{sur}} \circ \mathfrak{K}_\infty \subset \mathcal{I}^{\mathrm{sur}}$, and an ideal and its surjective hull produce the same class of compact sets [1, p. 79], we have

$$\mathcal{K}_{(\mathcal{L}_\infty^{\mathrm{adj}})^{\mathrm{dual}} \circ \mathfrak{K}_\infty} = \mathcal{K}_{\mathcal{I}^{\mathrm{sur}} \circ \mathfrak{K}_\infty} \subset \mathcal{K}_{\mathcal{I}^{\mathrm{sur}}} = \mathcal{K}_{\mathcal{I}} = \mathcal{K}_{\mathcal{N}},$$

and the proof is complete. ∎

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